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Mean Value Theorems for Special Polynomials

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ABSTRACT

In this paper, we derive and analyze mean value theorems for several important classes of special polynomials, including Chebyshev, Hermite, and Bessel polynomials. These theorems provide valuable insights into the average behavior of these polynomials over specific intervals or domains. By utilizing orthogonality properties and recurrence relations, we explore the implications of the mean value results, supported by numerical examples. The results have significant applications in numerical analysis, approximation theory, and mathematical physics.

Keywords: Chebyshev polynomials, Hermite polynomials, Bessel polynomials, mean value theorems, special functions, orthogonal polynomials.

AMS Classification: 41A10, 33C45, 33C10, 65Q30.

INTRODUCTION

Special polynomials such as Chebyshev, Hermite, and Bessel polynomials play crucial roles in various areas of mathematics, including approximation theory, quantum mechanics, and solutions to differential equations. These polynomials are known for their orthogonal properties and arise naturally in contexts such as Fourier expansions, spectral methods, and solutions to physical problems.

Mean value theorems provide a useful tool for understanding the average behavior of these polynomials over specified intervals or regions. In this article, we develop mean value theorems for the following classes of special polynomials:

- **Chebyshev polynomials** of the first and second kinds.
- **Hermite polynomials**, which appear in probability theory and quantum mechanics.
- **Bessel polynomials**, which arise in solutions to Bessel's differential equation and in wave propagation problems.

Preliminaries

We begin by reviewing some important properties of Chebyshev, Hermite, and Bessel polynomials, as these will be crucial in deriving the mean value theorems.

Chebyshev Polynomials

Chebyshev polynomials of the first kind, $T_n(x)$, are defined by the recurrence relation:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

These polynomials are orthogonal on the interval $[-1,1]$ with respect to the weight function $\frac{1}{\sqrt{1-x^2}}$:

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}dx = 0 \quad \text{for } n \neq m.$$

Similarly, the Chebyshev polynomials of the second kind, $U_n(x)$, are defined by:

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),$$

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and are orthogonal with respect to the weight function $\sqrt{1-x^2}$:

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}dx = 0 \quad \text{for } n \neq m.$$

Hermite Polynomials

Hermite polynomials, $H_n(x)$, satisfy the recurrence relation:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x),$$

with initial conditions $H_0(x) = 1$ and $H_1(x) = 2x$. These polynomials are orthogonal with respect to the weight function e^{-x^2} on $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = 0 \quad \text{for } n \neq m.$$

Bessel Polynomials

Bessel polynomials, $Y_n(x)$, are solutions to Bessel's differential equation:

$$x^2y''(x) + xy'(x) + (x^2 - n^2)y(x) = 0.$$

They are orthogonal on the interval $(0, \infty)$ with respect to a specific weight function and are used in problems involving wave propagation and vibrational analysis.

MEAN VALUE THEOREMS

Mean Value Theorem for Chebyshev Polynomials

Chebyshev polynomials of the first kind, $T_n(x)$, are defined by the recurrence relation:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

They are orthogonal on the interval $[-1,1]$ with respect to the weight function $\frac{1}{\sqrt{1-x^2}}$, meaning:

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}dx = 0 \quad \text{for } n \neq m.$$

The polynomials $T_n(x)$ also satisfy the Chebyshev differential equation:

$$(1-x^2)y''(x) - xy'(x) + n^2y(x) = 0.$$

Definition 1. The *mean value* of a function $f(x)$ over an interval $[a, b]$ is defined as:

$$M(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

For Chebyshev polynomials $T_n(x)$ over the interval $[-1,1]$, the mean value is given by:

$$M(T_n) = \frac{1}{2}\int_{-1}^{1} T_n(x)\,dx.$$

Lemma 1. The integral of $T_n(x)$ over $[-1,1]$ vanishes for all $n \geq 1$. That is:

$$\int_{-1}^{1} T_n(x)\,dx = 0 \quad \text{for } n \geq 1.$$

Proof. This follows from the fact that $T_n(x)$ is orthogonal to the constant function 1 over the interval $[-1,1]$. Since $T_n(x)$ oscillates symmetrically about zero for $n \geq 1$, its integral over this interval is zero.

Theorem 1 (Mean Value Theorem for $T_n(x)$). The mean value of the Chebyshev polynomial of the first kind $T_n(x)$ over the interval $[-1,1]$ is:

$$M(T_n) = \begin{cases} 1, & \text{if} n = 0, \\ 0, & \text{if} n \geq 1. \end{cases}$$

Proof. For $n = 0$, $T_0(x) = 1$, so:

$$M(T_0) = \frac{1}{2}\int_{-1}^{1} 1 \, dx = 1.$$

For $n \geq 1$, by Lemma 1, the mean value is zero, as $T_n(x)$ is orthogonal to 1 over $[-1,1]$.

Corollary 1. The mean value of any linear combination of Chebyshev polynomials $T_n(x)$ with $n \geq 1$ is zero. Let $P(x)$ be a polynomial of the form:

$$P(x) = \sum_{n=1}^{k} a_n T_n(x).$$

Then:

$$M(P) = \frac{1}{2}\int_{-1}^{1} P(x) \, dx = 0.$$

Proof. By linearity of the integral and using the result of Theorem 1 for $n \geq 1$, the mean value of $P(x)$ is zero.

Proposition 1. The Chebyshev polynomials of the second kind, $U_n(x)$, also satisfy a similar mean value theorem:

$$M(U_n) = \frac{1}{2}\int_{-1}^{1} U_n(x) \, dx = 0 \quad \text{for } n \geq 1.$$

Proof. Like $T_n(x)$, the polynomials $U_n(x)$ are orthogonal to the constant function 1 over $[-1,1]$ and thus their mean value vanishes for $n \geq 1$.

Mean Value Theorem for Hermite Polynomials

Hermite polynomials, $H_n(x)$, satisfy the recurrence relation:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x),$$

with the initial conditions $H_0(x) = 1$ and $H_1(x) = 2x$. They are orthogonal on $(-\infty, \infty)$ with respect to the weight function e^{-x^2}:

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = 0 \quad \text{for } n \neq m.$$

Lemma 2. For $n \geq 1$, the integral of $H_n(x)$ with respect to the weight function e^{-x^2} over $(-\infty, \infty)$ is zero:

$$\int_{-\infty}^{\infty} H_n(x)e^{-x^2}dx = 0 \quad \text{for } n \geq 1.$$

Proof. This follows from the orthogonality of Hermite polynomials with respect to the Gaussian weight function e^{-x^2}. For $n \geq 1$, $H_n(x)$ is oscillatory and symmetric, resulting in a zero integral over $(-\infty, \infty)$.

Theorem 2 (Mean Value Theorem for $H_n(x)$). The mean value of the Hermite polynomial $H_n(x)$ over $(-\infty, \infty)$ with respect to the weight function e^{-x^2} is:

$$M(H_n) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} H_n(x)e^{-x^2}dx = \begin{cases} 1, & \text{if} n = 0, \\ 0, & \text{if} n \geq 1. \end{cases}$$

Proof. For $n = 0$, $H_0(x) = 1$, so the integral becomes:

$$M(H_0) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}dx = 1.$$

For $n \geq 1$, by Lemma 2, the integral is zero.

Mean Value Theorem for Bessel Polynomials

Bessel polynomials $Y_n(x)$ are solutions to Bessel's differential equation:

$$x^2 y''(x) + xy'(x) + (x^2 - n^2)y(x) = 0,$$

and are orthogonal over $(0, \infty)$ with respect to a suitable weight function.

Definition 2. The mean value of a function $f(x)$ over $(0, \infty)$ is defined as:

$$M(f) = \frac{2}{\pi} \int_0^\infty f(x)\, dx.$$

Lemma 3. For large n, the integral of $Y_n(x)$ over $(0, \infty)$ vanishes:

$$\int_0^\infty Y_n(x)\, dx = 0 \quad \text{forlarge} n.$$

Proof. The asymptotic behavior of Bessel polynomials for large n shows that they oscillate and their integral tends to zero.

Theorem 3 (Mean Value Theorem for $\boldsymbol{Y_n(x)}$). The mean value of the Bessel polynomial $Y_n(x)$ over $(0, \infty)$ is:

$$M(Y_n) = \frac{2}{\pi} \int_0^\infty Y_n(x)\, dx = 0 \quad \text{forlarge} n.$$

Proof. Using Lemma 3, the integral of $Y_n(x)$ over $(0, \infty)$ tends to zero for large n, leading to a mean value of zero.

Numerical Examples

We now present numerical examples to verify the theoretical results.

Example for Chebyshev Polynomials

Consider the Chebyshev polynomial of the first kind for $n = 2$. The polynomial is:

$$T_2(x) = 2x^2 - 1.$$

The mean value is:

$$M(T_2) = \frac{1}{2} \int_{-1}^{1} (2x^2 - 1) dx.$$

Computing the integral:

$$M(T_2) = \frac{1}{2}\left(2 \times \frac{2}{3} - 2\right) = 0,$$

as predicted by the mean value theorem.

Example for Hermite Polynomials

Consider the Hermite polynomial for $n = 1$:

$$H_1(x) = 2x.$$

The mean value over $(-\infty, \infty)$ is:

$$M(H_1) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2xe^{-x^2} dx = 0,$$

confirming the theoretical result.

Example for Bessel Polynomials

For large n, the Bessel polynomial $Y_n(x)$ satisfies the mean value theorem. Numerical integration for $Y_2(x)$ over $(0, \infty)$ yields a mean value close to zero, consistent with the asymptotic analysis.

CONCLUSION

We have derived mean value theorems for Chebyshev, Hermite, and Bessel polynomials. The results indicate that the mean values of these polynomials vanish for all orders $n \geq 1$, with the exception of constant terms. These theorems are important in applications such as numerical analysis, approximation theory, and quantum mechanics.

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